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AN ARBITRARILY-ORIENTED PLANE CRACK TERMINATING AT THE INTERFACE BETWEEN DISSIMILAR PIEZOELECTRIC MATERIALS

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Abstract—The plane problem of a crack terminating at the interface of a bimaterial piezoelectric, and loaded on its faces, is treated. Emphasis is placed on how to transform this problem into a non-homogeneous Hilbert problem. To make the derivation tractable, the concept of the axial conjugate is introduced and related to the complex conjugate. The angle between the crack line and the interface may be arbitrary. Copyright © 1996 Elsevier Science Ltd.

1. INTRODUCTION

The study of cracks within piezoelectric materials is of paramount importance for many electroelastic micromechanics models and numerical fracture mechanics, because piezoceramic materials often contain many visible cracks prior to their employment. The existence of these defects greatly affects the electric, dielectric, piezoelectric, elastic and mechanical properties of piezoceramics (Okazaki, 1985). But this area of study has received relatively little attention in the literature. This is probably due to the complexity of the constituent equations for the materials, which are inherently anisotropic and involve a large number of material constants. Parton (1976) has considered the problem of a finite crack at the interface between two piezoelectric materials subjected to a far field uniform tension. Sosa and Pak (1990) developed a three dimensional solution for a semi-infinite crack in a piezoelectric material. More recently, Pak (1992) investigated the electroelastic fields and the energy release rate for a finite crack by way of the method of distributed dislocations and electric dipoles. Kuo and Barnett (1991) and Suo *et al.* (1992) solved the boundary value problems of electroelastic materials with interface cracks. Most of the above studies concentrated on the singularities at the tips of an interface crack. However, the problem of a crack terminating at, and at an arbitrary angle to, the interface between two piezoelectric materials does not seem to have been studied. In the following sections, the explicit solutions to the problem of an arbitrarily-oriented crack terminating at the interface between dissimilar piezoelectric materials will be derived by using the extended Stroh formalism (Barnett and Lothe, 1975) and the concept of the axial conjugate. The present problem will be transformed into a special case of the Hilbert problem and then its solution may be easily written out.

2. GENERAL SOLUTION TO THE GOVERNING EQUATIONS OF LINEAR PIEZOELECTRICS

In this section, the extended Stroh formalism (Barnett and Lothe, 1975) used to treat crack problems in dissimilar piezoelectric materials is reviewed. Throughout this paper, the shorthand notation introduced by Barnett and Lothe (1975) and rectangular Cartesian coordinates (x_1, x_2, x_3) are used. Lower case Latin subscripts will always range from 1 to 3, upper case Latin subscripts will range from 1 to 4, and the summation convention will be used for repeating subscripts unless it is otherwise indicated.

Consider a 2-D electroelastic problem in which all fields depend only on in-plane coordinates. In the absence of body forces and free charges, the basic equations used are as follows (Barnett and Lothe, 1975; Pak, 1992):

$$\Pi_{iJ,i} = 0 \tag{1}$$

$$\Pi_{iJ} = E_{iJKm} U_{K,m} \tag{2}$$

where

$$U_K = \begin{cases} u_k & K = 1, 2, 3 \\ \phi & K = 4 \end{cases} \tag{3}$$

$$\Pi_{iJ} = \begin{cases} \sigma_{ij} & i, J = 1, 2, 3 \\ D_i & i = 1, 2, 3, J = 4 \end{cases} \tag{4}$$

$$E_{iJKm} = \begin{cases} C_{ijkm} & J, K = 1, 2, 3 \\ e_{mij} & J = 1, 2, 3; \quad K = 4 \\ e_{ikm} & K = 1, 2, 3; \quad J = 4 \\ -\varepsilon_{im} & J = K = 4 \end{cases} \tag{5}$$

and u_k, ϕ, σ_{ij} and D_i are, respectively, elastic displacements, electric potential, stresses and electric displacements. C_{ijkm}, ε_{ij} and e_{ijk} are, respectively, elastic moduli, dielectric constants and piezoelectric constants. The solutions to eqn (1) may be expressed as (Pak, 1992)

$$\mathbf{U} = \mathbf{a}\mathbf{f}(z) \tag{6}$$

$$z = x_1 + px_2 \tag{7}$$

where $\mathbf{U} = \{u_1\, u_2\, u_3\, \phi\}^T$, $\mathbf{f}$ is an arbitrary function vector to be determined, $\mathbf{a}$ and p are obtained from

$$[\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}]\mathbf{a} = 0 \tag{8}$$

and $\mathbf{Q}$, $\mathbf{R}$ and $\mathbf{T}$ are 4×4 matrices:

$$Q_{IK} = E_{1IK1}, \quad R_{IK} = E_{1IK2}, \quad T_{IK} = E_{2IK2}. \tag{9}$$

This is an eigenvalue problem consisting of four equations, for which a nontrivial $\mathbf{a}$ exists if p is a root of the determinantal equation

$$\|\mathbf{Q} + p(\mathbf{R} + \mathbf{R}^T) + p^2\mathbf{T}\| = 0. \tag{10}$$

Since (10) admits no real root, the eight roots, $p_1, \ldots, p_8$, form four conjugate pairs (Suo *et al.*, 1992). Let $\mathrm{Im}(p_I) > 0$ with associated vectors $\mathbf{a}_I$, and define

$$\mathbf{f}(z) = \{f_1(z_1) \;\; f_2(z_2) \;\; f_3(z_3) \;\; f_4(z_4)\}^T \tag{11}$$

$$\mathbf{A} = [\mathbf{a}_1 \;\; \mathbf{a}_2 \;\; \mathbf{a}_3 \;\; \mathbf{a}_4] \tag{12}$$

where $z_I = x_1 + p_I x_2$.

The solution for $\mathbf{U}$ and $\boldsymbol{\sigma}_i$ is then given by (Suo *et al.*, 1992)

$$\mathbf{U} = \mathbf{A}\mathbf{f}(z) + \overline{\mathbf{A}\mathbf{f}}(\bar{z}) \tag{13}$$

$$\boldsymbol{\sigma}_i = \mathbf{B}_i\mathbf{g}(z) + \bar{\mathbf{B}}_i\bar{\mathbf{g}}(\bar{z}) \quad (i = 1, 2) \tag{14}$$

where $\boldsymbol{\sigma}_i = \{\sigma_{i1}\ \sigma_{i2}\ \sigma_{i3}\ D_i\}^{\mathrm{T}}$, $\mathbf{g}(z) = \mathrm{d}\mathbf{f}(z)/\mathrm{d}z$, the overbar denotes the complex conjugate, and $\mathbf{B}_i$ in (14) is a 4×4 matrix:

$$(\mathbf{B}_i)_{KM} = (E_{iKJ1} + p_M E_{iKJ2})A_{JM} \quad (M \text{ not summed}). \tag{15}$$

3. AN ARBITRARILY-ORIENTED CRACK ENDING AT THE INTERFACE

Consider an inclined crack of length $2a$ terminating at the interface between two dissimilar anisotropic piezoelectric half spaces (Fig. 1). The bonded interface coincides with the x_1-axis, and one of the crack tips touches the interface at $x_1 = 0$. The material constants of the inhomogeneous composite solid are expressed as

$$E_{iJKm} = \begin{cases} E^{U}_{iJKm} & x_2 > 0 \\ E^{L}_{iJKm} & x_2 < 0 \end{cases} \tag{16}$$

where, here and subsequently, the superscripts "U" and "L" are associated with the upper material and the lower material respectively. Let $-\mathbf{t}(s)$ denote the traction and surface charges acting over the crack faces. The boundary conditions for the problem with continuous $\mathbf{U}$ and σ_2 across the bimaterial interface are then

$$\mathbf{U}^{L}(x_1, 0) = \mathbf{U}^{U}(x_1, 0) \tag{17}$$

$$\boldsymbol{\sigma}^{L}_2(x_1, 0) = \boldsymbol{\sigma}^{U}_2(x_1, 0) \tag{18}$$

$$\mathbf{U}(x_1, x_2) \to 0 \quad |x_1^2 + x_2^2| \to \infty \tag{19}$$

$$\boldsymbol{\sigma}^{L}_i(s, \alpha^-)n_i = \boldsymbol{\sigma}^{U}_i(s, \alpha^+)n_i = -\mathbf{t}(s) \quad |s| < a \tag{20}$$

where $\boldsymbol{\sigma}_i$ $(i = 1, 2)$ are vectors defined in the last section, $(n_1, n_2) = (-\sin\alpha, \cos\alpha)$ and α and s are coordinates local to the crack, shown in Fig. 1.

Inserting (13) into (17), one has

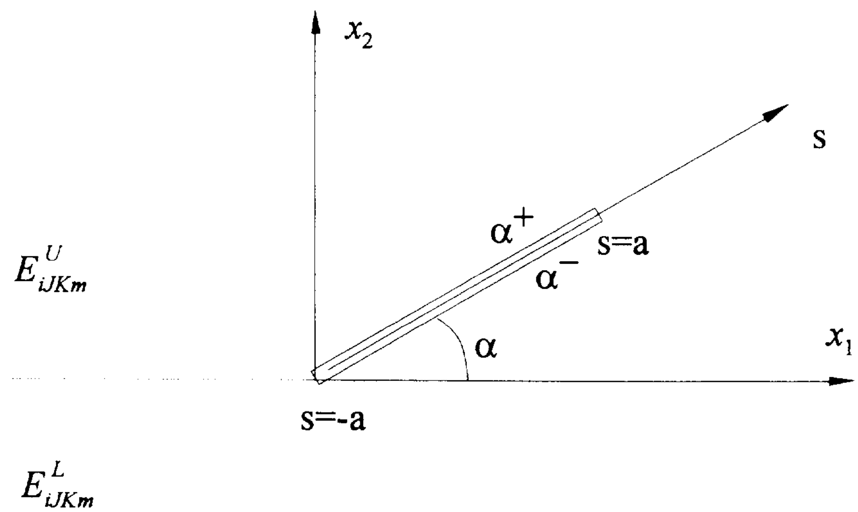

Fig. 1. Geometry for an inclined crack.

$$\mathbf{A}^U \mathbf{f}^U(x_1) - \bar{\mathbf{A}}^L \bar{\mathbf{f}}^L(x_1) = \mathbf{A}^L \mathbf{f}^L(x_1) - \bar{\mathbf{A}}^U \bar{\mathbf{f}}^U(x_1) \quad x_2 = 0. \tag{21}$$

It is obvious that the left-hand side of (21) is a function, analytic in the upper half-plane and the right-hand side is another function, analytic in the lower half-plane. Hence, put

$$\mathbf{A}^U \mathbf{f}^U(z) - \bar{\mathbf{A}}^L \bar{\mathbf{f}}^L(z) = \psi(z) \quad z \in U \tag{22}$$

$$\mathbf{A}^L \mathbf{f}^L(z) - \bar{\mathbf{A}}^U \bar{\mathbf{f}}^U(z) = \psi(z) \quad z \in L \tag{23}$$

where the function $\psi(z)$ is analytic in the whole plane, ensures that both sides of (21) become satisfied identically. Similarly, from (14), (18) and (19)

$$\mathbf{B}_2^U \mathbf{g}^U(z) = \bar{\mathbf{B}}_2^L \bar{\mathbf{g}}^L(\bar{z}) \quad z \in U \tag{24}$$

$$\mathbf{B}_2^L \mathbf{g}^L(z) = \bar{\mathbf{B}}_2^U \bar{\mathbf{g}}^U(\bar{z}) \quad z \in L. \tag{25}$$

If the eigenvalues p are all distinct, $\mathbf{B}_i^L$ and $\mathbf{B}_i^U$ $(i = 1, 2)$ are non-singular. In this case, eqns (24) and (25) may be solved for $\mathbf{g}^L(z)$ and $\mathbf{g}^U(z)$, which yields

$$\mathbf{g}^U(z) = (\mathbf{B}_2^U)^{-1} \bar{\mathbf{B}}_2^L \bar{\mathbf{g}}^L(\bar{z}) \quad z \in U \tag{26}$$

$$\mathbf{g}^L(z) = (\mathbf{B}_2^L)^{-1} \bar{\mathbf{B}}_2^U \bar{\mathbf{g}}^U(\bar{z}) \quad z \in L. \tag{27}$$

Differentiating (22) and noting (26) yields

$$\mathbf{C}\mathbf{g}^U(z) = \psi'(z) \quad z \in U \tag{28}$$

where the prime denotes the derivative with respect to the associated argument, and

$$\mathbf{C} = \mathbf{A}^U - \bar{\mathbf{A}}^L (\bar{\mathbf{B}}_2^L)^{-1} \mathbf{B}_2^U. \tag{29}$$

To study the property of matrix $\mathbf{C}$, define

$$\mathbf{H} = \mathbf{Y}^U + \bar{\mathbf{Y}}^L \tag{30}$$

$$\mathbf{Y} = i\mathbf{A}\mathbf{B}_2^{-1} \tag{31}$$

where $\mathbf{H}$ is Hermitian (Suo *et al.*, 1992). Hence

$$\mathbf{C} = -i\mathbf{H}\mathbf{B}_2^U. \tag{32}$$

It can be seen from (32) that $\mathbf{C}$ has the same rank as that of $\mathbf{H}$. In general, $\mathbf{H}$ is non-singular (Suo *et al.*, 1992). Thus

$$\mathbf{g}^U(z) = \mathbf{C}^{-1}\psi'(z) \quad z \in U. \tag{33}$$

The present problem is now transformed into a non-homogenous Hilbert problem. To make the derivation tractable, the concept of axial conjugate is introduced in the following

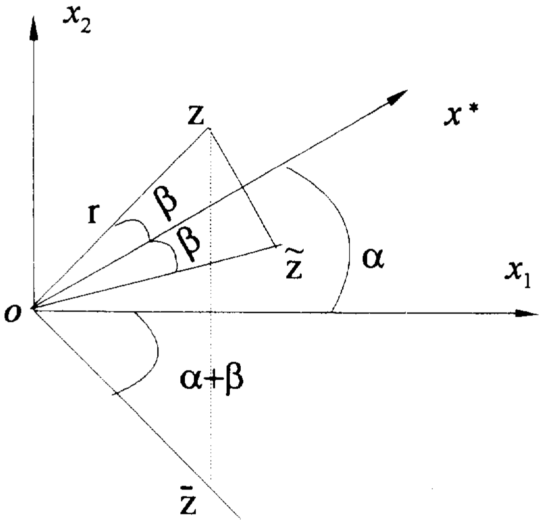

Fig. 2. The definitions of $\tilde{z}$ and $\bar{z}$.

way. Let $\tilde{z}$ denote the axial conjugate of z about the x^*-axis (Fig. 2). It can be seen from Fig. 2 that

$$\tilde{z} = re^{i(\alpha-\beta)} = re^{-i(\alpha+\beta)+2i\alpha} = e^{2i\alpha}\bar{z} \tag{34}$$

which is the relation between axial and complex conjugates. As a consequence, (14) may be rewritten in the form

$$\boldsymbol{\sigma}_i = \mathbf{B}_i\mathbf{g}(z) + e^{-2i\alpha}\bar{\mathbf{B}}_i\bar{\mathbf{g}}(\tilde{z}). \tag{35}$$

The substitution of (35) into (20) and noting (21), (22) and (33) yields

$$\mathbf{G}\boldsymbol{\psi}'^{-}(s) + \bar{\mathbf{G}}\boldsymbol{\psi}'^{+}(s) = -e^{i\alpha}\mathbf{t}(s) \quad |s| \leqslant a \tag{36}$$

where

$$s = \frac{z}{\cos\alpha + p\sin\alpha} - a$$

and

$$\mathbf{G} = (\mathbf{B}_2^t\cos\alpha - \mathbf{B}_1^t\sin\alpha)e^{i\alpha}\mathbf{C}^{-1}. \tag{37}$$

Following the technique of Clements (1971), the problem can be transformed into

$$\mathbf{S}\boldsymbol{\psi}'^{-} + \boldsymbol{\lambda}^*\mathbf{S}\boldsymbol{\psi}'^{+}(s) = -e^{i\alpha}\mathbf{N}\mathbf{t}(s) \quad |s| \leqslant a \tag{38}$$

in which $\boldsymbol{\lambda}^* = \mathrm{diag}[\lambda_1\ \lambda_2\ \lambda_3\ \lambda_4]$, and $\mathbf{N}$ is obtained from

$$(\bar{\mathbf{G}} - \lambda\mathbf{G})\mathbf{N} = 0 \tag{39}$$

with

$$|\bar{\mathbf{G}} - \lambda\mathbf{G}| = 0 \tag{40}$$

and the matrix $\mathbf{S}$ is evaluated from

$$\mathbf{S} = \mathbf{N}\mathbf{G}. \tag{41}$$

Noting that the rank of **S** is important later, a proof is presented to show that, when the four roots λ_l of (40) are all distinct and the matrix **C** is in full rank, the matrix **S** is non-singular. First, it can be seen from (39) that **N** is non-singular, by the assumption that all the λ_l are distinct. In this case **S** has the same rank as that of **G**, from (41). Next it is proved that **G** is non-singular if **C** is in full rank. Thus, defining

$$\mathbf{P}^{\zeta} = \mathrm{diag}[p_1^{\zeta} \;\; p_2^{\zeta} \;\; p_3^{\zeta} \;\; p_4^{\zeta}] \tag{42}$$

so that (Qu and Bassani, 1989)

$$\mathbf{B}_1^{\zeta} = -\mathbf{B}_2^{\zeta}\mathbf{P}^{\zeta}. \tag{43}$$

From (37), one has

$$\mathbf{G} = \mathbf{B}_2^{\zeta}(\cos\alpha\mathbf{I} + \sin\alpha\mathbf{P}^{\zeta})e^{i\alpha}\mathbf{C}^{-1} \tag{44}$$

where **I** is the unit matrix. Since $\mathbf{B}_2^{\zeta}$ is in full rank by the previous assumption, **G** has the same rank as that of **C**. Therefore it is clear from (41) that **S** is non-singular, by the assumption that **C** is in full rank and all the λ_l are distinct.

The appropriate solution to (38) may now be given as (see Clements, 1971)

$$\mathbf{S}\psi'(z) = \frac{-e^{i\alpha}}{2\pi i}\mathbf{X}(s(z))\int_{-a}^{a}\mathbf{X}^*(s(z),\xi)\mathbf{N}\mathbf{t}(\xi)\,d\xi \tag{45}$$

where

$$\begin{aligned}
\mathbf{X}(s) &= \mathrm{diag}[X_1(s) \;\; X_2(s) \;\; X_3(s) \;\; X_4(s)] \\
\mathbf{X}^*(s,\xi) &= \mathrm{diag}[y_1(s,\xi) \;\; y_2(s,\xi) \;\; y_3(s,\xi) \;\; y_4(s,\xi)] \\
X_K(s) &= (s+a)^{-m_K}(s-a)^{m_K-1} \\
m_K &= \frac{1}{2\pi i}\log\lambda_K \\
y_K(s,\xi) &= \frac{1}{X_K^+(\xi)(\xi-s)}.
\end{aligned} \tag{46}$$

If $|\mathbf{S}| \neq 0$ then

$$\psi'(z) = \frac{-e^{i\alpha}}{2\pi i}\mathbf{S}^{-1}\mathbf{X}(s(z))\int_{-a}^{a}\mathbf{X}^*(s(z),\xi)\mathbf{N}\mathbf{t}(\xi)\,d\xi. \tag{47}$$

The right-hand side of (47) may be evaluated numerically since all terms of it are known functions. For example, one may calculate them by an integration formula of Gaussian type, for which (47) becomes

$$\psi'(z) = \frac{-e^{i\alpha}}{2\pi i}\mathbf{S}^{-1}\mathbf{X}(s(z))\sum_{j=1}^{n}\mathbf{X}^*(s(z),\xi_j)\mathbf{N}\mathbf{t}(\xi_j)W_j \tag{48}$$

where W_j ($j = 1, 2, \ldots n$) are the weighting constants of the related integration formula. By inserting into (35), one obtains

$$\boldsymbol{\sigma}_q^F = -2\mathrm{Re}\frac{-e^{i\alpha}}{2\pi i}\mathbf{B}_q^F\mathbf{C}^{-1}\mathbf{S}^{-1}\mathbf{X}(s(z))\sum_{j=1}^{n}\mathbf{X}^*(s(z),\xi_j)\mathbf{Nt}(\xi_j)W_j \quad (F = L, U) \tag{49}$$

where $\boldsymbol{\sigma}_i^L = \{\sigma_{i1}^L\ \sigma_{i2}^L\ \sigma_{i3}^L\ D_i^L\}^T$, σ_{ij}^L stands for the stress in the upper material, and others can be defined similarly.

4. CRACK TIP SINGULARITIES OF Π_{ij}

In order to study the crack tip singularities of stress and electric displacement, the polar coordinates (r, θ) centered at the tips with $\theta = 0$ along the crack line are introduced. An asymptotic form is used to discuss the order of singularity for Π_{ij} at the two crack tips:

(a) Crack tip at $x_1 = x_2 = 0$.

Let r be the distance from the tip. In this case the variable z can be expressed as

$$z = r(\cos(\theta+\alpha)+p\sin(\theta+\alpha)) \tag{50}$$

the singularity near the tip can be obtained by taking the asymptotic limit of (49) as $r \to 0$

$$\boldsymbol{\sigma}_q^L(r,\theta) = \mathrm{Re}\left[\mathbf{F}_{1q}^L\left\langle r^{-m_K}\left(\frac{\cos(\theta+\alpha)+p\sin(\theta+\alpha)}{\cos\alpha+p\sin\alpha}\right)^{-m_K}(-2a)^{m_K-1}\right\rangle\mathbf{F}_{2q}^L\right] \tag{51}$$

where $F = L, U$ were defined in (49), $q = 1, 2$, and

$$\langle L(m_K)\rangle = \mathrm{diag}[L(m_1)L(m_2)L(m_3)L(m_4)] \tag{52}$$

$$\mathbf{F}_{1q}^L = \frac{i}{\pi}e^{i\alpha}\mathbf{B}_q^L\mathbf{C}^{-1}\mathbf{S}^{-1} \tag{53}$$

$$\mathbf{F}_{2q}^L = \sum_{\beta=1}^{n}\mathbf{X}^*(-a,\xi_\beta)\mathbf{Nt}(\xi_\beta)W_\beta. \tag{54}$$

Hence, there are four modes of singularities at the tip touching the interface: r^{-m_K} $(K = 1, 2, 3, 4)$, where m_K are defined in $(46)_4$.

(b) Crack tip at $x_1 = 2a\cos\alpha$ and $x_2 = 2a\sin\alpha$.

Again let r be the distance from the tip. In this case the variable z can be expressed as

$$z-2a(\cos\alpha+p\sin\alpha) = r(\cos(\theta+\alpha)+p\sin(\theta+\alpha)). \tag{55}$$

The order of singularity for Π_{ij} near the tip can be obtained by taking the asymptotic limit of (49) as $r \to 0$.

$$\boldsymbol{\sigma}_q^L(r,\theta) = \mathrm{Re}\left[\mathbf{F}_{1q}^L\left\langle r^{m_K-1}\left(\frac{\cos(\theta+\alpha)+p\sin(\theta+\alpha)}{\cos\alpha+p\sin\alpha}\right)^{m_K-1}(-2a)^{-m_K}\right\rangle\mathbf{F}_{2q}^L\right] \tag{56}$$

which shows that the orders of the singularity in the stress and electric displacement fields at the tip are r^{m_K-1} $(K = 1, 2, 3, 4)$.

In addition, if the two materials become identical and $\alpha = 0$, it follows from (44) that

$$\mathbf{G} = (\mathbf{AB}_2^{-1} - \bar{\mathbf{A}}\bar{\mathbf{B}}_2^{-1})^{-1}. \tag{57}$$

It is obvious that $\bar{\mathbf{G}} = -\mathbf{G}$. Therefore, the roots λ_j of (40) are equal to -1. Substituting this result into $(46)_4$, one sees that $m_K = 1/2$ for all K. Thus, the matrix $\mathbf{S}$ may be singular. In this case we can directly solve (36) for $\psi'(z)$, which yields

$$\psi'(z) = \frac{\mathbf{G}^{-1}}{2\pi i}\int_{-a}^{a}\frac{\mathbf{t}(\tau)\,\mathrm{d}\tau}{\tau - z}. \tag{58}$$

Further, if $\alpha \neq 0$ we may also prove that $m_K = 1/2$ through use of a coordinate system local to the crack line and an appropriate transformation given in Yu and Qin (1996). For the sake of conciseness we omit those details, which are tedious and algebraic here.

5. NUMERICAL ILLUSTRATION

For convenience we only consider an inclined crack terminating at the interface between transversely isotropic materials. The upper and lower materials are assumed to be the *PZT-5H* and the *PZT-5*, respectively. The material constants for the two materials are given in the Appendix.

Equation (10), for the case of a transversely isotropic material, reduces to

$$(c_{44}p^2 + c_{66})(a_0p^6 + a_1p^4 + a_2p^2 + a_3) = 0 \tag{59}$$

with

$$\begin{aligned}
a_0 &= -c_{33}e_{35}^2 - \varepsilon_{33}c_{33}c_{66} \\
a_1 &= \varepsilon_{33}[c_{13}^2 + 2c_{13}c_{66} - c_{33}c_{11}] + (e_{13}+e_{35})[2e_{35}(c_{13}+c_{66}) - c_{66}(e_{13}+e_{35})] \\
&\quad - e_{35}(2c_{33}e_{11} + c_{66}e_{35}) - \varepsilon_{11}c_{33}c_{66} \\
a_2 &= \varepsilon_{11}[c_{13}^2 + 2c_{13}c_{66} - c_{33}c_{11}] + (e_{13}+e_{35})[2e_{11}(c_{13}+c_{66}) - c_{11}(e_{13}+e_{35})] \\
&\quad - e_{11}(c_{33}e_{11} + 2c_{66}e_{35}) - \varepsilon_{33}c_{11}c_{66} \\
a_3 &= -c_{66}(\varepsilon_{11}c_{11} + e_{11}^2)
\end{aligned}$$

in which the well-known two index notation has been adopted (Nye, 1957). c_{ij} and e_{ij} are the reduced material constants obtained by using the following convention: replace ij or kl by p and q, where i, j, k, l take the values of 1–3, and p, q assume the values 1–6 according to the following:

ij or kl	11	22	33	23 or 32	31 or 13	12 or 21
p or q	1	2	3	4	5	6

In accordance to this representation it follows that $c_{pq} = C_{ijkl}$, $e_{ip} = e_{ikl}$, for $i, j, k, l = 1$–3, $p, q = 1$–6.

The solutions of (59) for *PZT-5H* and *PZT-5* are

$$\begin{aligned}
p_1^U &= -0.1735+0.9318i, & p_2^U &= 0.1735+0.9318i, \\
p_3^U &= 0.9337i, & p_4^U &= 0.9972i, \\
p_1^L &= -0.2033+0.8813i, & p_2^L &= 0.2033+0.8813i, \\
p_3^L &= 0.9280i, & p_4^L &= 0.9620i.
\end{aligned}$$

Table 1. The values of m_K vs angle α

α	0	5	15	25	35	45	90
$\mathrm{Re}(m_1)$	0.5000	0.5304	0.3493	0.2447	0.1382	0.0299	0.5000
$\mathrm{Re}(m_2)$	0.5417	0.4457	0.4004	0.3553	0.3060	0.2533	0.0678
$\mathrm{Re}(m_3)$	0.4599	0.4451	0.5167	0.5077	0.5022	0.4977	0.0696
$\mathrm{Re}(m_4)$	0.5000	0.4230	0.4169	0.3615	0.3060	0.2504	0.5000
$\mathrm{Im}(m_1 \times 10^3)$	0.00	−0.7225	0.0903	0.1734	0.0565	−0.446	0.0002
$\mathrm{Im}(m_2 \times 10^3)$	−0.3466	2.147	−0.0037	−0.0684	−0.1400	−2.368	0.0013
$\mathrm{Im}(m_3 \times 10^3)$	0.1517	−7.569	−2.312	−3.558	−1.944	−5.445	0.0021
$\mathrm{Im}(m_4 \times 10^3)$	0.0	0.0	0.0	0.0	0.0	0.0	0.0

The variations of the singularity order m_K with the angle α are presented in Table 1.

It can be seen from Table 1 that the order of singularity in the stress and electric displacement fields depends strongly on the inclined angle α. However, the numerical results indicate that the values of $\mathrm{Im}(m_k)$ are very small for the material combination *PZT*-5*H* and *PZT*-5.

6. CONCLUSIONS

The plane problem of a crack that terminates at the interface between two piezoelectric materials has been treated. The study shows that this problem can be transformed into a non-homogeneous Hilbert problem by way of the concept of the axial conjugate. The numerical results indicate that the order of singularity in the traction-charge field at the crack tips depends strongly on the inclined angle α.

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APPENDIX

(1) Material properties for *PZT*-5*H* (Pak, 1992)

$c_{22} = c_{33} = 126\,\mathrm{GPa}$,
$c_{23} = 55\,\mathrm{GPa}$,
$c_{12} = c_{13} = 53\,\mathrm{GPa}$,
$c_{11} = 117\,\mathrm{GPa}$,
$c_{55} = c_{66} = 35.3\,\mathrm{GPa}$,
$c_{44} = \frac{1}{2}(c_{33} - c_{23}) = 35.5\,\mathrm{GPa}$,
$e_{12} = e_{13} = -6.5\,\mathrm{C/m^2}$,

$e_{11} = 23.3\,\mathrm{C/m^2}$,
$e_{35} = e_{26} = 17\,\mathrm{C/m^2}$,
$\varepsilon_{33} = \varepsilon_{22} = 151 \times 10^{-10}\,\mathrm{C/Vm}$,
$\varepsilon_{11} = 130 \times 10^{-10}\,\mathrm{C/Vm}$.

(2) Material properties for *PZT*-5 (Dunn and Taya, 1993):
$c_{22} = c_{33} = 121\,\mathrm{GPa}$,
$c_{23} = 75.4\,\mathrm{GPa}$,
$c_{12} = c_{13} = 75.2\,\mathrm{GPa}$,
$c_{11} = 111\,\mathrm{GPa}$,
$c_{55} = c_{66} = 21.1\,\mathrm{GPa}$,
$c_{44} = \frac{1}{2}(c_{33} - c_{23}) = 22.8\,\mathrm{GPa}$,
$e_{12} = e_{13} = -5.4\,\mathrm{C/m^2}$,
$e_{11} = 15.8\,\mathrm{C/m^2}$,
$e_{35} = e_{26} = 12.3\,\mathrm{C/m^2}$,
$\varepsilon_{33} = \varepsilon_{22} = 81.07 \times 10^{-10}\,\mathrm{C/Vm}$,
$\varepsilon_{11} = 73.46 \times 10^{-10}\,\mathrm{C/Vm}$.

It should be noted that the indices in the Appendix of Pak (1992) and in Table 1 of Dunn and Taya (1993) have been changed due to the different coordinate systems used. In our study the x_1-axis is chosen to be the poling direction, and the crack line is in the x_1–x_2 plane.